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No. 898

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UNDER SHEAR AND AXIAL STRESSES

By A. Kromm

Luftfahrtforschung
Vol. 15, No. 10/11, October 10, 1938
Verlag von R. Oldenbourg, München und Berlin

Washington
June 1939

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THE LIMIT OF STABILITY OF A CURVED PLATE STRIP
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By A. Kromm

For an infinitely long plate strip of constant curvature there is set up the buckling condition for combined shear and axial stresses. Assumptions are made of the vanishing of the moments acting on the longitudinal edges and of the vanishing of the tangential displacements u and the transverse stresses σ_y (longitudinal stiffeners nonresistant to bending in the circumferential direction). The buckling condition is evaluated for the case of pure shear and an approximate formula is derived for the critical stresses for large curvatures. In connection with the discussion of the methods and results there is defined the range of validity of the buckling condition set up.

I. INTRODUCTION AND STATEMENT OF THE PROBLEM

The stressed skin covering of fuselages and wings is sectioned off by longitudinal stiffeners into panels of small curvature by these panels constituting curved plate strips. If a membrane type stress condition is produced in these structural parts the equilibrium configuration reaches the limit of stability at a definite load. Since each plate strip represents only a small portion of the stressed-skin structure the state of stress may be assumed as homogeneous before buckling sets in and there then arises the important problem of determining the critical loading limit for shear or compressive stress or for the two combined.

The stability problems of a curved plate strip may be

* "Die Stabilitätsgrenze eines gekrümmten Plattenstreifens bei Beanspruchung durch Schub- und Längskräfte." Luftfahrtforschung, vol. 15, no. 10/11, Oct. 10, 1938, pp. 517-26.

classified both as to the type of loading and the end conditions. Since the resistance to buckling of the longitudinal stiffeners is considerably higher than that of the plate, a rigid support in the radial direction must be assumed at the edges (with the limiting cases of simple (no moment) or clamped support). The tangential displacements, on account of the large extensional stiffness of the stiffeners as well as the radial displacements are to be set equal to zero at the edges. For the tangent to the plate at right angles to the edge, the following limiting conditions are possible: The vanishing of the normal displacements (flexurally rigid edge stiffeners) or the vanishing of the normal stresses (edge stiffeners without flexural resistance in the plate surface).

A number of the problems here discussed have previously been treated. The critical stress in the case of compressive stresses has been determined by S. Timoshenko (reference 1) for an infinitely long plate strip with constant curvature under the assumption of simply supported longitudinal edges. Leggett (reference 2) has investigated the critical shear stress of a plate strip of small curvature for simply supported and clamped (flexurally stiff) edges. His numerical evaluation of the buckling condition set up is valid, however, for only plates with very small curvature. In view of airplane construction requirements, however, his computations must be extended to larger curvatures.

In the present investigation the buckling condition is set up for an infinitely long plate strip of constant curvature for the case of moment-free support at edges for combined shear and axial loading. The buckling condition is evaluated for the case of pure shear stress and for larger curvatures an approximate formula is derived. The evaluation of the buckling condition for the general case of combined shear and axial stresses will shortly be published. In contrast to what was done by Leggett we shall start with the complete fundamental equations for the displacements. With the aid of an expression by Ritz for the buckling functions the factors neglected by Leggett will then be justified for a certain range of plate dimensions.

II. GENERAL PRINCIPLES

Let there be given an infinitely long plate strip of width b , thickness s , and constant radius of curvature r (fig. 1). The x, y, z coordinates are taken respectively in the directions of the generators, along the circumference and along the radius, the x axis coinciding with one of the edges of the strip. Corresponding to the coordinates are the displacements u, v, w , of the middle surface, which carry the plate from the initial equilibrium condition to the neighboring condition on reaching the limit of stability. To produce these displacements an amount of work A_1 in deforming the plate (potential of internal forces) and an amount of work A_2 against the external forces is required. Denoting the negative product of the external forces by the displacements of their points of application by $V = -A_2$ the change in the total potential energy due to the virtual displacements u, v, w , is equal to

$$\delta \Pi = A_1 + V \quad (1)$$

The linear components of the displacements u, v, w , in $\delta \Pi$ must vanish since in the initial state the plate strip was in equilibrium. Below the stability limit the square components Q of $\delta \Pi$ are positive (minimum of potential energy in the stable equilibrium state). At the stability limit at least one set of u, v, w must be possible for which the square components Q vanish but none for which they are negative. From this minimum property it follows that for all admissible variations it is necessary that

$$\delta Q = 0 \quad (2)$$

(See reference 3.)

To set up the expression for the deformation work it is necessary to find the relation between the plate deformation and the values of the displacements. The work done in deforming the plate is made up of the work done by the basic stresses, uniformly distributed over the plate thickness, in deforming the plate middle surface and in the deformation work of the additional buckling stresses. In the first portion the deformation magnitudes enter linearly while in the second they enter squared.

Since the expression for the strain energy must also contain the square terms in the displacements u, v, w , the squares of the displacements must be taken into account in the deformations of the plate middle surface.

Let t_1, t_2, t_3 be the unit vectors in the direction of the generators, the circumference, and the radius, respectively, at any point of the middle surface of the plate (fig. 2). The displacement v will here be assumed along the tangent and not along the circumference. Let the position vector of the middle surface of the plate before the deformation be $\underline{R} = \underline{R}_0$. The square of the length of the line element before the deformation is then

$$d s_0^2 = \left(\frac{\partial \underline{R}_0}{\partial x} dx + \frac{\partial \underline{R}_0}{\partial y} dy \right)^2 = (t_1 dx + t_2 dy)^2 = dx^2 + dy^2$$

After the deformation

$$\underline{R} = \underline{R}_0 + u t_1 + v t_2 + w t_3$$

$$\frac{\partial \underline{R}}{\partial x} = \underline{R}_x = (1 + u_x) t_1 + v_x t_2 + w_x t_3$$

$$\frac{\partial \underline{R}}{\partial y} = \underline{R}_y = u_y t_1 + \left(1 + v_y + \frac{w}{r} \right) t_2 + \left(w_y - \frac{v}{r} \right) t_3$$

and

$$\begin{aligned} ds^2 = & dx^2 [1 + 2 u_x + u_x^2 + v_x^2 + w_x^2] \\ & + dy^2 \left[1 + 2 \left(v_y + \frac{w}{r} \right) + u_y^2 + \left(v_y + \frac{w}{r} \right)^2 + \left(w_y - \frac{v}{r} \right)^2 \right] \\ & + 2 dx dy \left[u_y + v_x + u_x u_y + v_x \left(v_y + \frac{w}{r} \right) + w_x \left(w_y - \frac{v}{r} \right) \right] \end{aligned}$$

For the changes in the coefficients of the square of the linear element of the middle surface of the plate we then have

$$\left. \begin{aligned} \bar{\gamma}_{11} &= 2 u_x + u_x^2 + v_x^2 + w_x^2 \\ \bar{\gamma}_{22} &= 2 \left(v_y + \frac{w}{r} \right) + u_y^2 + \left(v_y + \frac{w}{r} \right)^2 + \left(w_y - \frac{v}{r} \right)^2 \\ \bar{\gamma}_{12} &= u_y + v_x + u_x u_y + v_x \left(v_y + \frac{w}{r} \right) + w_x \left(w_y - \frac{v}{r} \right) \end{aligned} \right\} (3.1-3)$$

The geometrical significance of these deformation magnitudes is obtained from their relations with the extensional and angular strains of the surface. We have (reference 4)

$$\left. \begin{aligned} \bar{\epsilon}_x &= \sqrt{1 + \bar{\gamma}_{11}} - 1 = u_x + \frac{1}{2} v_x^2 + \frac{1}{2} w_x^2 \\ \bar{\epsilon}_y &= \sqrt{1 + \bar{\gamma}_{22}} - 1 = v_y + \frac{w}{r} + \frac{1}{2} u_y^2 + \frac{1}{2} \left(w_y - \frac{v}{r} \right)^2 \\ \sin \bar{\gamma}_{xy} &= \frac{\bar{\gamma}_{12}}{\sqrt{1 + \bar{\gamma}_{11}} \sqrt{1 + \bar{\gamma}_{22}}} = \frac{\bar{\gamma}_{12}}{(1 + \bar{\epsilon}_x)(1 + \bar{\epsilon}_y)} \end{aligned} \right\} (4.1-3)$$

In the linear portions the magnitude $\bar{\gamma}_{12}$ is equal to $\bar{\gamma}_{xy}$; the values $\bar{\gamma}_{11}$ and $\bar{\gamma}_{22}$ differ from the corresponding strains ϵ_x and ϵ_y by the factor 2.

For the deformations at the distance z from the middle surface there are obtained under the usual assumptions of plate bending theory the following relations

$$\left. \begin{aligned} \epsilon_x &= u_x - z w_{xx} \\ \epsilon_y &= v_y + \frac{w}{r+z} - z \frac{r}{r+z} w_{yy} \\ \gamma_{xy} &= \frac{r}{r+z} u_y + \frac{r+z}{r} v_x - z \left(1 + \frac{r}{r+z} \right) w_{xy} \end{aligned} \right\} (5.1-3)$$

(The detailed derivation of these and the following relations are found in reference 5.) Since these magnitudes enter only into the strain energy of the additional (buckling) stresses only the linear portions need be taken into account. By Hooke's law the additional stresses in the plate after buckling are found to be

$$\left. \begin{aligned} \sigma_x &= \frac{E}{1 - \nu^2} (\epsilon_x + \nu \epsilon_y) \\ \sigma_y &= \frac{E}{1 - \nu^2} (\epsilon_y + \nu \epsilon_x) \\ \tau &= \frac{E}{2(1 + \nu)} \gamma_{xy} \end{aligned} \right\} \text{--- (6)}$$

These equations, integrated over the plate thickness, taking account of the trapezoidal form of the plate element in the cross section, give the following resultant forces and moments (fig. 3)

$$\left. \begin{aligned} N_x &= D \left(u_x + \nu v_y + \nu \frac{w}{r} \right) - \frac{K}{r} w_{xx} \\ N_y &= D \left(\nu u_x + v_y + \frac{w}{r} \right) + \frac{K}{r} \left(w_{yy} + \frac{w}{r^2} \right) \\ N_{xy} &= D \frac{1 - \nu}{2} (u_y + v_x) + \frac{K}{r} \frac{1 - \nu}{2} \left(\frac{v_x}{r} - w_{xy} \right) \\ N_{yx} &= D \frac{1 - \nu}{2} (u_y + v_x) + \frac{K}{r} \frac{1 - \nu}{2} \left(\frac{u_y}{r} - w_{xy} \right) \\ M_x &= K \left(w_{xx} + \nu w_{yy} - \frac{u_x}{r} - \nu \frac{v_y}{r} \right) \\ M_y &= K \left(\nu w_{xx} + w_{yy} + \frac{w}{r^2} \right) \\ M_{xy} &= K (1 - \nu) \left(w_{xy} - \frac{v_x}{r} \right) \\ M_{yx} &= K (1 - \nu) \left(w_{xy} + \frac{u_y}{2r} - \frac{v_x}{2r} \right) \end{aligned} \right\} \text{--- (7. 1-8)}$$

where $D = \frac{E s}{1 - \nu^2}$ is the axial rigidity of the plate and

$K = \frac{E s^3}{12(1 - \nu^2)}$ the flexural rigidity.

The deformation work of the additional stresses per unit area of the middle surface is

$$a_1 = \frac{1}{2} \int_{-s/2}^{+s/2} (\sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau \gamma_{xy}) \left(1 + \frac{z}{r}\right) dz \quad (8)$$

Substituting equations (6) there is obtained

$$a_1 = \frac{1}{2} \frac{E}{1-\nu^2} \int_{-s/2}^{+s/2} \left[(\epsilon_x + \epsilon_y)^2 - 2(1-\nu) \left(\epsilon_x \epsilon_y - \frac{1}{4} \gamma_{xy}^2 \right) \right] \left(1 + \frac{z}{r}\right) dz \quad (8a)$$

For the infinitely long plate strip the displacements must be periodic along the longitudinal direction. We shall denote the half-wave length by l . For the strain energy of the additional stresses over the plate width b and a complete wave length $2l$ there follows from (8a):

$$A_1 = \frac{1}{2} \frac{E}{1-\nu^2} \int_0^{2l} \int_0^b \int_{-s/2}^{+s/2} \left[(\epsilon_x + \epsilon_y)^2 - 2(1-\nu) \left(\epsilon_x \epsilon_y - \frac{1}{4} \gamma_{xy}^2 \right) \right] \left(1 + \frac{z}{r}\right) dx dy dz \quad (9)$$

The basic (original) stresses are made up of the membrane shear forces τs and the longitudinal forces σs , where positive values of σ denote tensile stresses. Since the moment of the additional stresses about the normal to the plate

$$r(N_{xy} - N_{yx}) + M_{yx}$$

vanishes identically, as can be readily seen by substituting the deformation values from equations (7), the moments of the basic stresses at the deformed element must be also in equilibrium. (See reference 5, p. 200.) For the shear forces on the deformed shell element, we must therefore substitute $\tau s(1 + \bar{\epsilon}_y)dy$ and $\tau s(1 + \bar{\epsilon}_x)dx$. For the longitudinal forces, the assumption may be made that they act along the direction of the edges. The strain energy of the basic shear forces per unit area of surface (see fig. 4) is obtained as

$$a_2 = \tau s (1 + \bar{\epsilon}_x)(1 + \bar{\epsilon}_y) \sin \bar{\gamma}_{xy} \quad (10)$$

With the aid of the third of equations (4) there is obtained

$$a_2 = \tau s \bar{\gamma}_{12} \quad (10a)$$

The strain energy of the longitudinal forces per unit area is obtained as

$$a_3 = \sigma s \bar{\epsilon}_x \quad (11)$$

Integrating over the plate width and a wave length, there is obtained from equations (10a) and (11) for the strain energy of the basic stresses

$$\begin{aligned} A_2 + A_3 = \tau s \int_0^{2l} \int_0^b & \left[u_y + v_x + u_x u_y + v_x \left(v_y + \frac{w}{r} \right) \right. \\ & \left. + w_x \left(w_y - \frac{v}{r} \right) \right] dx dy + \sigma s \int_0^{2l} \int_0^b \left[u_x + \frac{1}{2}(v_x^2 + w_x^2) \right] dx dy \end{aligned} \quad (12)$$

The potential of the external forces is

$$\begin{aligned} V = - \tau s \int_0^{2l} (u_{y=b} - u_{y=0}) dx - \tau s \int_0^b (v_{x=2l} - v_{x=0}) dy \\ - \sigma s \int_0^b (u_{x=2l} - u_{x=0}) dy \end{aligned} \quad (13)$$

As may be easily seen from equations (12) and (13) the linear strain terms in the sum cancel each other (condition of equilibrium). In the change in potential energy

$$\delta \Pi = A_1 + V = A_1 + A_2 + A_3 + V$$

there remain only the quadratic portions Q . By substituting the displacements (5) in equation (9), there is obtained

$$\begin{aligned}
Q = & \frac{D}{2} \int_0^a \int_0^b \left\{ \left(u_x + v_y + \frac{w}{r} \right)^2 - 2(1 - \nu) \left[(u_x v_y - u_y v_x) \right. \right. \\
& \left. \left. - \frac{1}{4} (u_y - v_x)^2 + u_x \frac{w}{r} \right] \right\} dx dy + \frac{K}{2} \int_0^a \int_0^b \left\{ \left(w_{xx} + w_{yy} + \frac{w}{r^2} \right)^2 \right. \\
& \left. - 2 \left(u_x + v_y + \frac{w}{r} \right) \frac{w_{xx}}{r} - 2(1 - \nu) \left[(w_{xx} w_{yy} - w_{xy}^2) \right. \right. \\
& \left. \left. + \frac{1}{2} (v_x - u_y) \frac{w_{xy}}{r} + \left(v_x \frac{w_{xy}}{r} - v_y \frac{w_{xx}}{r} \right) - \frac{u_y^2 + 3v_x^2}{4r^2} \right] \right\} dx dy \\
& + \tau s \int_0^a \int_0^b \left[u_x v_y + v_x \left(v_y + \frac{w}{r} \right) + w_x \left(w_y - \frac{v}{r} \right) \right] dx dy \\
& + \sigma s \frac{1}{2} \int_0^a \int_0^b (v_x^2 + w_x^2) dx dy \tag{14}
\end{aligned}$$

For simply supported longitudinal edges $w = 0$ and $w_{yy} = 0$ at $y = 0$ and $y = b$. Furthermore, the tangential displacement u must also vanish at these edges. Making use of these end conditions and the periodicity of the displacements along the longitudinal direction and integrating in the first term of the sum partially with respect to x and the second term partially with respect to y , it is readily seen that the portions

$$u_x v_y - u_y v_x, w_{xx} w_{yy} - w_{xy}^2, \text{ and } v_x \frac{w_{xy}}{r} - v_y \frac{w_{xx}}{r}$$

do not contribute to the strain energy. From the minimum condition $\delta Q = 0$ we then have

$$\begin{aligned}
-\frac{1}{D} \delta Q_u = & \int_0^a \int_0^b \left[u_{xx} + \frac{1-v}{2} u_{yy} + \frac{1+v}{2} v_{xy} + v \frac{w_x}{r} \right. \\
& + k \left(r \frac{1-v}{2} w_{xyy} - r w_{xxx} + \frac{1-v}{2} u_{yy} \right) \\
& \left. + 2 q_1 u_{xy} \right] \delta u \, dx \, dy = 0 \quad (15.1)
\end{aligned}$$

$$\begin{aligned}
-\frac{1}{D} \delta Q_v = & \int_0^a \int_0^b \left[v_{yy} + \frac{1-v}{2} v_{xx} + \frac{1+v}{2} u_{xy} + \frac{w_y}{r} \right. \\
& + k \left(-r \frac{3-v}{2} w_{xxy} + \frac{3(1-v)}{2} v_{xx} \right) + 2 q_1 \left(v_{xy} + \frac{w_x}{r} \right) \\
& \left. + q_2 v_{xx} \right] \delta v \, dx \, dy - \int_0^a \left\{ (v_y + q_1 v_x) \delta v \right\} \bigg|_0^b dx = 0 \quad (15.2)
\end{aligned}$$

$$\begin{aligned}
\frac{r}{D} \delta Q_w = & \int_0^a \int_0^b \left[v u_x + v_y + \frac{w}{r} + \frac{k}{r} \left(r^4 \Delta \Delta w + 2 r^2 w_{yy} \right. \right. \\
& \left. \left. + w + r^3 \frac{1-v}{2} u_{xyy} - r^3 u_{xxx} - r^3 \frac{3-v}{2} v_{xxy} \right) \right. \\
& \left. + 2 q_1 (v_x - r w_{xy}) - q_2 r w_{xx} \right] \delta w = 0 \quad (15.3)
\end{aligned}$$

where

$$k = \frac{K}{D r^2} = \frac{1}{12} \left(\frac{s}{r} \right)^2, \quad q_1 = \frac{rs}{D} (1 - v^2) \frac{1}{E}$$

and

$$q_2 = \frac{os}{D} = (1 - v^2) \frac{\sigma}{E}$$

Since the equations (15) thus set up must vanish for each variation δu , δv , and δw , the expressions in brackets under the integral signs are to be set equal to zero. This leads to three linear differential equations for the displacements u , v , w , which equations may also be derived from a consideration of the stability of a shell element. For longitudinal members that are flexurally yielding in the circumferential direction δv is not equal to zero at $y = 0$ and $y = b$ and from the boundary integral of equation (15₂), there is then obtained as the boundary condition for the displacement at $y = 0$ and $y = b$.

$$v_y + q_1 v_x = 0$$

Considering equation (7.2), it may be seen that the normal stress and the component of the shear force in the circumferential direction at the deformed edges $y = 0$ and $y = b$ must be in equilibrium. The complete boundary conditions at $y = 0$ and $y = b$ are therefore

$$\left. \begin{aligned} w &= 0 \\ w_{yy} &= 0 \\ u &= 0 \\ v_y + q_1 v_x &= 0 \end{aligned} \right\} \text{-----} (16.1-4)$$

In an exact determination of the critical stresses, the solutions of the differential equations in (15) must be substituted in the boundary conditions (16). There are then obtained eight homogeneous equations (four at each edge) for u , v , and w . The problem is then to find the smallest value of q_1 for given q_2 (or conversely) for which a nonidentically vanishing solution for the displacements is possible. The differential equations may be solved by assuming for the displacements the expressions

$$u = A \sin \frac{1}{r} (\lambda x - m y)$$

$$v = B \sin \frac{1}{r} (\lambda x - m y)$$

$$w = C \cos \frac{1}{r} (\lambda x - m y)$$

The continuation of the method leads, however, to computational difficulties which, like those met in the case of a twisted unstiffened circular cylindrical shell of finite length (reference 5, p. 203), exceed reasonable bounds. The computation will here be carried out with the aid of a Ritz expression in the form already employed by Leggett for the strip of small curvature and by E. Seydel for the flat strip (reference 6).

III. SETTING UP OF THE BUCKLING CONDITION WITH THE AID OF A RITZ SUBSTITUTION

The Ritz method consists in approximating the displacements by using a combination of partial expressions in which undetermined values enter linearly, each expression satisfying the boundary conditions. The values are then so determined that the approximating expression, substituted in equation (14), reduces the latter to a minimum. The calculus of variations problem then goes over into an ordinary minimum problem. If the minimum conditions for the quadratic terms in the change of potential energy are set up before the integration (method of Galerkin, reference 7) there is obtained again the system of equations (15) where, for example, there is to be substituted

$\delta u = \sum_m \frac{\partial u}{\partial A_m} \delta A_m$ where A_m are the undetermined values entering in the expression for u . Since

only known functions now stand in the integrands, the integration can be performed and on the left-hand sides of the equations, there are then obtained polynomials in the variations δA_m , etc., the undetermined values themselves occurring in the coefficients. Since the equations must be satisfied for each system of values δA_m , the coefficients must vanish separately and this leads to a system of equations for computing the undetermined values. This system of equations will have a solution for the displacements other than zero only if the determinant of its coefficients vanishes. There is thus obtained the required condition for the reaching of the stability limit.

The oblique buckling waves that are obtained in the presence of shear are represented in the expressions for the displacements by two partial waves for each displacement shifted in phase by a half-wave length in the longitudinal direction:

$$\begin{aligned}
 u &= \cos \frac{\pi x}{l} \sum_m A_{1m} \sin \frac{m \pi y}{b} \\
 &\quad - \sin \frac{\pi x}{l} \sum_m A_{2m} \sin \frac{m \pi y}{b} \\
 v &= \sin \frac{\pi x}{l} \sum_m B_{1m} \cos \frac{m \pi y}{b} \\
 &\quad + \cos \frac{\pi x}{l} \sum_m B_{2m} \cos \frac{m \pi y}{b} \\
 w &= \sin \frac{\pi x}{l} \sum_m C_{1m} \sin \frac{m \pi y}{b} \\
 &\quad + \cos \frac{\pi x}{l} \sum_m C_{2m} \sin \frac{m \pi y}{b}
 \end{aligned}
 \tag{17.1-3}$$

with $m = 1, 2, 3, \dots$. The boundary conditions (16.1-3) are exactly satisfied by the chosen expressions, while boundary condition (16.4) is satisfied except for the term $q_1 v_x$. As may be seen from equation (17.2) the derivatives of v with respect to x and y for the range $(0 < y < b)$ are of the same order of magnitude. Since $q_1 = \frac{T}{E} (1 - \nu^2) \ll 1$, it follows that an exact satisfying of the boundary condition (16.4) would require in the expression chosen for v only a slight correction which is dispensed with here.

As will be shown below, in the first partial waves we require only the even terms in m and in the second only the odd terms (or conversely). The number of terms taken gives the degree of the approximation for the buckling functions and for the critical stresses. As the computation for the case of pure shear shows, the series of approximations for the critical stresses converges very rapidly. The summation for the partial waves in the transverse direction of the plate strip needs to be extended only to $m = 4$.

The functions thus chosen for the displacements u , v , w are now substituted in equations (15.1-3). Since the boundary condition (equation 16.4) is not exactly satisfied, the boundary integral in equation (15.2) will yield a small contribution. Making use of

$$\left. \begin{aligned} \int_0^{2l} \sin^2 \frac{\pi x}{l} dx &= \int_0^{2l} \cos^2 \frac{\pi x}{l} dx = l, \\ \int_0^{2l} \sin \frac{\pi x}{l} \cos \frac{\pi x}{l} dx &= 0 \\ \int_0^b \sin \frac{n\pi y}{b} \cos \frac{m\pi y}{b} dy &= \begin{cases} \frac{b}{\pi} \frac{2n}{n^2 - m^2} & \text{für } n \pm m \text{ ungerade} \\ 0 & \text{für } n \pm m \text{ gerade} \end{cases} \\ \int_0^b \sin \frac{n\pi y}{b} \sin \frac{m\pi y}{b} dy &= \int_0^b \cos \frac{n\pi y}{b} \cos \frac{m\pi y}{b} dy = \begin{cases} 0 & \text{für } n \neq m \\ \frac{b}{2} & \text{für } n = m \end{cases} \end{aligned} \right\} \quad (18)$$

there is obtained the following system of equations:

$$\left. \begin{aligned} A_{1n} a_n + q_1 \beta \frac{8}{\pi} \sum_m A_{2m} \frac{nm}{n^2 - m^2} + B_{1n} f_n - C_{1n} \frac{b}{\pi r} g_n &= 0 \\ A_{2n} a_n - q_1 \beta \frac{8}{\pi} \sum_m A_{1m} \frac{nm}{n^2 - m^2} + B_{2n} f_n - C_{2n} \frac{b}{\pi r} g_n &= 0 \end{aligned} \right\} \quad (19_{1-2})$$

$$\left. \begin{aligned} A_{1n} f_n + B_{1n} b_n + q_1 \beta \frac{8}{\pi} \sum_m B_{2m} \frac{1}{2} \frac{n^2 + m^2}{n^2 - m^2} - C_{1n} \frac{b}{\pi r} h_n - q_1 \beta \frac{8}{\pi} \sum_m C_{2m} \frac{m}{n^2 - m^2} &= 0 \\ A_{2n} f_n + B_{2n} b_n - q_1 \beta \frac{8}{\pi} \sum_m B_{1m} \frac{1}{2} \frac{n^2 + m^2}{n^2 - m^2} - C_{2n} \frac{b}{\pi r} h_n + q_1 \beta \frac{8}{\pi} \sum_m C_{1m} \frac{m}{n^2 - m^2} &= 0 \end{aligned} \right\} \quad (20_{1-2})$$

$$\left. \begin{aligned} -A_{1n} \frac{b}{\pi r} g_n - B_{1n} \frac{b}{\pi r} h_n - q_1 \beta \frac{8}{\pi} \sum_m B_{2m} \frac{n}{n^2 - m^2} + C_{1n} \frac{b^2}{\pi^2 r^2} c_n + q_1 \beta \frac{8}{\pi} \sum_m C_{2m} \frac{nm}{n^2 - m^2} &= 0 \\ -A_{2n} \frac{b}{\pi r} g_n - B_{2n} \frac{b}{\pi r} h_n + q_1 \beta \frac{8}{\pi} \sum_m B_{1m} \frac{n}{n^2 - m^2} + C_{2n} \frac{b^2}{\pi^2 r^2} c_n - q_1 \beta \frac{8}{\pi} \sum_m C_{1m} \frac{nm}{n^2 - m^2} &= 0 \end{aligned} \right\} \quad (21_{1-2})$$

where, for briefness, there has been set

$$\left. \begin{aligned} a_n &= \beta^2 + \frac{1-\nu}{2} n^2 (1+k), \\ b_n &= n^2 + \frac{1-\nu}{2} \beta^2 (1+3k) + q_2 \beta^2, \\ c_n &= 1 + k \frac{\pi^4 r^4}{b^4} \left[(n^2 + \beta^2)^2 - 2 \frac{b^2}{\pi^2 r^2} n^2 + \frac{b^4}{\pi^4 r^4} \right] + q_2 \frac{\pi^2 r^2}{b^2} \beta^2, \\ f_n &= \frac{1+\nu}{2} n \beta, \quad g_n = \beta \left[\nu + k \frac{\pi^2 r^2}{b^2} \left(\beta^2 - \frac{1-\nu}{2} n^2 \right) \right], \\ h_n &= n \left(1 + k \frac{\pi^2 r^2}{b^2} \frac{3-\nu}{2} \beta^2 \right) \text{ und } \beta = \frac{b}{l} \end{aligned} \right\} \quad (22)$$

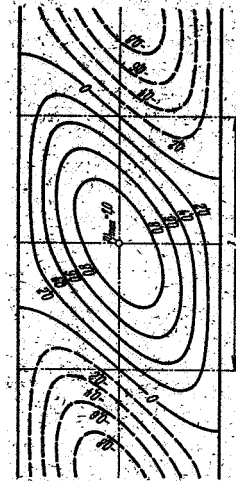


Figure 9.- Contour lines of the buckling surface for pure shear stress of a flat plate strip, $\beta = b/l = 0.8$.

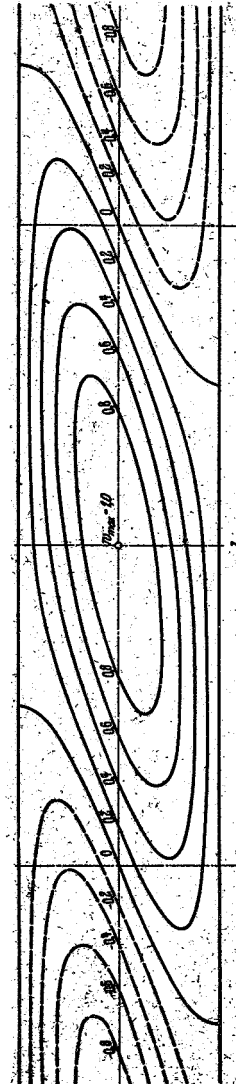


Figure 8.- Contour lines of the buckling surface in the case of pure shear stressing of a curved plate strip with $b/\pi r = 1/4$, $b/s = 50$, $\beta = b/l = 0.1$.

In the summation terms m is to be taken so that $n \pm m$ is an odd number (see equation (18)); i.e., for n even, m must be odd and conversely. The structure of the system of equations is shown by the scheme of figure 5 which represents the part of equations (21.₁₋₂) with the unknowns C_{1n} , C_{2n} . It may be seen that the entire system breaks up into two independent partial systems whose coefficients are distinguished by the heavy and light dashes. From equations (21) it is further evident that the coefficients of C_{1n} in equation (21.₁) are equal to those of C_{2n} in equation (21.₂) whereas the right upper region in figure 5 and the lower left region contain equal and opposite coefficients. If therefore the unknowns C_{11} , C_{13}, C_{22} , C_{24} ... are computed from one partial system (light dashes) the solution of the other partial system can be found from the relations

$$\left. \begin{array}{ll} C_{12} = C_{22} & C_{21} = -C_{11} \\ C_{14} = C_{24} & C_{23} = -C_{13} \\ \dots\dots\dots & \dots\dots\dots \\ \dots\dots\dots & \dots\dots\dots \end{array} \right\} \text{----- (23)}$$

The same holds true for the A's and B's since the entire system (19-21) is constructed in exactly the same manner as the portion shown in figure 5.

Substituting the equations (23) and the corresponding expressions for A and B in (17), it is seen that the two partial systems of figure 5 represent the same buckling form except for a phase displacement of $1/2$ so that it is sufficient to investigate one of them in which the sum of the two subscripts of the unknown is odd. The pairs of equations (19) (20) and (21) can then each be combined into one as follows:

$$\left. \begin{aligned}
 A_n a_n - q_1 \beta \frac{8}{\pi} \sum_m A_m \frac{n m}{n^2 - m^2} (-1)^m \\
 + B_n f_n - C_n \frac{b}{\pi r} g_n = 0 \\
 A_n f_n + B_n b_n - q_1 \beta \frac{8}{\pi} \sum_m B_m \frac{1}{2} \frac{n^2 + m^2}{n^2 - m^2} (-1)^m \\
 - C_n \frac{b}{\pi r} h_n + q_1 \beta \frac{8}{\pi} \frac{b}{\pi r} \sum_m C_m \frac{m}{n^2 - m^2} (-1)^m = 0 \\
 - A_n \frac{b}{\pi r} g_n - B_n \frac{b}{\pi r} h_n + \\
 + q_1 \beta \frac{8}{\pi} \frac{b}{\pi r} \sum_m B_m \frac{n}{n^2 - m^2} (-1)^m \\
 + C_n \frac{b^2}{\pi^2 r^2} c_n - q_1 \beta \frac{8}{\pi} \sum_m C_m \frac{n m}{n^2 - m^2} (-1)^m = 0
 \end{aligned} \right\} (24.1-3)$$

where n and m now run through both even and odd values.

For $n = 1, 2 \dots i$ and $m = 1, 2 \dots i$ (in each term $n \pm m$ is odd), there is obtained from the coefficients of the system of equations (24.1-3) a determinant symmetric with respect to its main diagonal, of the i^{th} order. The required buckling condition is found in the following manner. Let us consider the first two systems of equations of (24). For a finite number of terms $n = 1, 2 \dots i$ the undetermined values A and B can be expressed in terms of the C 's by Cramer's rule. From the consideration of the values a_n, b_n, f_n, g_n, h_n , and the summation terms, it may be seen that the determinant of the denominator is of the order of magnitude $i!^4$, and similarly the determinant of the numerator of A_n . The numerator determinant of B_n , however, is of the order of magnitude $\frac{1}{n} i!^4$. By substituting the obtained expressions in the system of equation (24.3) there is obtained a determinant of the coefficients of the i^{th} order. By

dividing through by the term $k \frac{\pi^4 r^4}{b^4} (n^2 + \beta^2)^2$ this determinant may be brought into a normal form:

$$\begin{vmatrix} 1 + c_{11} & c_{12} & c_{13} & \dots \\ c_{21} & 1 + c_{22} & c_{23} & \dots \\ c_{31} & c_{32} & 1 + c_{33} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \quad (25)$$

where the infinite series

$$c_{11} + c_{12} + c_{21} + c_{13} + c_{22} + c_{31} + \dots \quad (26)$$

converges absolutely, since n^{-2} is the highest power of n occurring in $c_{\alpha\beta}$. This, however, is a sufficient condition for the convergence of the determinant (25). The latter set equal to zero gives the buckling condition for the curved plate strip under combined shear and axial stresses.

In the elimination of A and B it is necessary to solve determinants of the 12 order. With certain restrictions for the plate dimensions, which, however are within the requirements of airplane construction, the computations may be considerably simplified. Equations (24.1-3) are first transformed as follows:

$$\left. \begin{aligned} A_n \left[a_n - \lambda \sum \frac{A_m}{A_n} \frac{n m}{n^2 - m^2} (-1)^m \right] + B_n f_n - C_n \frac{b}{\pi r} g_n &= 0 \\ A_n f_n + B_n \left[b_n - \lambda \sum \frac{B_m}{B_n} \frac{1}{2} \frac{n^2 + m^2}{n^2 - m^2} (-1)^m \right] \\ - C_n \frac{b}{\pi r} \left[h_n - \lambda \sum \frac{C_m}{C_n} \frac{m}{n^2 - m^2} (-1)^m \right] &= 0 \\ - A_n \frac{b}{\pi r} g_n - B_n \frac{b}{\pi r} \left[h_n - \lambda \sum \frac{B_m}{B_n} \frac{n}{n^2 - m^2} (-1)^m \right] \\ + C_n \left[\frac{b^2}{\pi^2 r^2} c_n - \lambda \sum \frac{C_m}{C_n} \frac{n m}{n^2 - m^2} (-1)^m \right] &= 0 \end{aligned} \right\} \quad (27.1-3)$$

where, for briefness, there has been set $\lambda = q_1 \beta \frac{8}{\pi}$. Introducing the values

$$p^* = \frac{4 \pi^2 K}{b^2 s} = \frac{\pi^2}{3} \frac{E}{1 - \nu^2} \frac{s^2}{b^2}$$

as reference values for the critical stresses and remembering that $q_1 = \frac{\tau}{E} (1 - \nu^2)$, there is obtained for λ the expression

$$\lambda = \frac{\tau}{p^*} \beta \frac{8 \pi}{3} \frac{s^2}{b^2} \quad (28)$$

The reference value p^* thus introduced is the critical compressive stress of a flat plate of width b for simply supported longitudinal edges. For the flat plate $\tau/p^* = 1.334$ and $\beta = 0.793$; for a curved plate τ/p^* increases with the curvature but the product $\frac{\tau}{p^*} \beta$ remains, as further computation will show, approximately constant. Thus λ on account of $s/b \ll 1$ is also a very small magnitude.

For the flat plate ($r = \infty$) equations (27.₁₋₂) separate from equation (27.₃). It is known that the vanishing of the brackets after C_n in equation (27.₃) gives the smallest value for λ and that $A_n \equiv B_n \equiv 0$. Also in the case of plates with small curvature it is assumed in a first approximation that the λ term in the bracket after C_n equation (27.₃) remains the essential buckling term of the system of equations. If the λ terms in equations (27.₁₋₂) and the small magnitudes k and q_2 are dropped then A_n and B_n may be expressed in terms of C_n from these two equations, as follows:

$$\left. \begin{aligned} A_n &= - \frac{b}{\pi r} \beta \frac{n^2 - \nu \beta^2}{(n^2 + \beta^2)^2} C_n \\ B_n &= \frac{b}{\pi r} n \frac{n^2 + (2 + \nu) \beta^2}{(n^2 + \beta^2)^2} C_n \end{aligned} \right\} \quad (29)$$

from which is obtained:

$$\left. \begin{aligned} \frac{A_m}{A_n} &= \frac{m^2 - v \beta^2}{n^2 - v \beta^2} \frac{(n^2 + \beta^2)^2}{(m^2 + \beta^2)^2} \frac{C_m}{C_n} \\ \frac{B_m}{B_n} &= \frac{m m^2 + (2 + v) \beta^2}{n n^2 + (2 + v) \beta^2} \frac{(n^2 + \beta^2)^2}{(m^2 + \beta^2)^2} \frac{C_m}{C_n} \end{aligned} \right\} \quad (30)$$

With the aid of these first approximations the ratios between the λ terms and the parts a_n , b_n , etc., in the coefficients will be set up as a check on the approximations made. If β^2 is neglected compared with n^2 or m^2 and in addition there are neglected (for $r \neq \infty$) the small magnitudes k and q_2 , the required ratios are found to be the following:

For A_n in (27.1)

$$\frac{2}{1 - v} \lambda \sum_m \frac{C_m}{C_n} \frac{1}{n^2 - m^2} \frac{n}{m} (-1)^m : 1$$

For B_n in (27.2)

$$\lambda \sum_m \frac{C_m}{C_n} \frac{1}{n^2 - m^2} \frac{1}{2} \left(\frac{n}{m} + \frac{m}{n} \right) (-1)^m : 1$$

For C_n in (27.2)

$$\lambda \sum_m \frac{C_m}{C_n} \frac{1}{n^2 - m^2} \frac{m}{n} (-1)^m : 1$$

For B_n in (27.3)

$$\lambda \sum_m \frac{C_m}{C_n} \frac{1}{n^2 - m^2} \frac{n}{m} (-1)^m : 1$$

For C_n in (27.3)

$$\lambda \sum_m \frac{C_m}{C_n} \frac{1}{n^2 - m^2} n m \frac{\pi^2 r^2}{b^2} (-1)^m : 1$$

(31.1-5)

For $r \rightarrow \infty$ ratio (31.5) will become infinite. As shown by equations (22), however, the k and q_2 term in c_n

is not to be neglected compared to 1, but conversely, and there is obtained in place of (31.₅)

$$\lambda \sum_m \frac{C_m}{C_n} \frac{n m}{n^2 - m^2} \frac{1}{\frac{\pi^2}{12} \frac{s^2}{b^2} (n^2 + \beta^2)^2 + \frac{\pi^2}{3} \frac{s^2}{b^2} \frac{\sigma}{p^*} \beta^2} (-1)^m : 1 \quad (32)$$

where the reference value p^* has also been introduced in the term with q_2 . The ratios (31.₁₋₄) are independent of the width of the plate. For sufficiently small curva-

ture ($\frac{\pi^2 r^2}{b^2} \gg 1$) these ratios are smaller than (31.₅) by

the factor $\pi^2 r^2 / b^2$ and in addition smaller than (32) by n^2 or m^2 in passing to the flat plate on account of

$\frac{s^2}{b^2} \ll 1$. There has thus been confirmed the assumption

made above that the λ term after C_n in equation (27.₃)

remains the essential buckling term also for the case of plate strips of small curvature. The magnitudes of the ratios thus set up are dependent of C_m/C_n . For values of n to which correspond large values of C_n (in the present case for $n = 1$ to 3, as will be shown with the aid of examples); i.e., for considerable portions of the system of equations the ratios (31.₁₋₄) are small and the first approximations for A_n and B_n , according to equations (29) are sufficiently accurate. For larger values of n , that is, smaller values of C_n , the accuracy will decrease but the corresponding buckling terms, as will be shown on evaluating the buckling determinant, are of small significance.

The determinant of the coefficients of (27) must be equal to zero for nonvanishing values of A_n , B_n , and C_n . If, in the course of the computation, the powers and products of k , q_2 and of terms with λ are neglected and for A_m/A_n and B_m/B_n their first approximations from (30) are substituted, we arrive at the following system of equations:

$$\begin{aligned}
C_n & \left\{ (1 - \nu^2) \beta^4 + k \frac{\pi^4 r^4}{b^4} \left[(n^2 + \beta^2)^4 \right. \right. \\
& - 2 \frac{b^2}{\pi^2 r^2} (n^6 + (4 - \nu) n^4 \beta^2 + 3 n^2 \beta^4 + \nu \beta^6) \\
& + \frac{b^4}{\pi^4 r^4} (n^4 + 2(2 - \nu) n^2 \beta^2 + (4 - 3 \nu^2) \beta^4) \left. \right] \\
& + q_2 \frac{\pi^2 r^2}{b^2} \beta^2 \left[(n^2 + \beta^2)^2 + \frac{b^2}{\pi^2 r^2} (n^2 + 2(1 + \nu) \beta^2) \right] \left. \right\} \\
& = q_1 \beta \frac{8}{\pi} \frac{\pi^2 r^2}{b^2} (n^2 + \beta^2)^2 \sum_m C_m \frac{n m}{n^2 - m^2} (-1)^m \left\{ 1 \right. \\
& + \frac{b^2}{\pi^2 r^2} \frac{1}{(m^2 + \beta^2)^2} \left[\frac{m^2 - \nu \beta^2}{n^2 - \nu \beta^2} \beta^2 \right. \\
& + \frac{m^2 + (2 + \nu) \beta^2}{n^2 + (2 + \nu) \beta^2} (n^2 + 2(1 + \nu) \beta^2) \frac{m^2 + n^2}{2 n^2} \\
& - \frac{(m^2 + \beta^2)^2}{(n^2 + \beta^2)^2} (n^2 + (2 + \nu) \beta^2) \\
& \left. \left. - \frac{m^2 + (2 + \nu) \beta^2}{n^2 + (2 + \nu) \beta^2} (n^2 + (2 + \nu) \beta^2) \right] \right\} \quad (33)
\end{aligned}$$

The convergence of the determinant of the coefficients in equations (33) may be easily proved if the equations are divided through by $k \frac{\pi^4 r^4}{b^4} (n^2 + \beta^2)^4$. There is then obtained the normal form (25) with the converging series (26). The left side of equation (33) is exact except for the neglected products and powers of k and q_2 and has the same form as for a circular, cylindrical shell under compression and without stiffeners.*

*(See reference 5, p. 196.) The circular, cylindrical shell under compression during buckling is divided up by the nodal lines of the buckled form in the x direction into curved plate strips with $w = w_{yy} = u = v_y = 0$ at the nodal lines; i.e., the same "boundary conditions" which are considered here. The exact computations of W. Flügge can therefore be used for the curved plate strip.

If we neglected the shear load terms in equations (27) with the exception of the important buckling term for small curvature strips at C_n in the third equation, there would remain on the right side of equations (33) only the 1 in the braces. Let us denote the other terms as correction terms and expand them, since β^2 is a small magnitude, in powers of β . We break off at the quadratic terms in β and obtain for the braces the following expression

$$\left\{ 1 - \frac{b^2}{\pi^2 a r^2} \left[\frac{1}{2} \left(\frac{1}{n^2} + \frac{1}{m^2} \right) + \frac{1}{2} \left(\frac{1}{n^2} - \frac{1}{m^2} \right)^2 \nu \beta^2 - \frac{1}{n^2 m^2} \beta^2 \right] \right\} \quad (34)$$

The above elimination process is sufficiently accurate if the correction terms (34) are small as compared with 1. For a plate strip with a central angle of 45° , i.e., with

$\frac{b}{\pi r} = \frac{1}{4}$, corresponding to a circular, cylindrical shell

with 8 longitudinal stiffeners, the correction terms for $n = 1$ and $m = 2$ ($n \pm m$ odd) on neglecting the small terms with β^2 amount to about -4 percent. On the left side of equation (33) the coefficients of k and q_2 (square brackets) are arranged in powers of $b/\pi r$. For

$\frac{b}{\pi r} = \frac{1}{4}$ the portions with $b^4/\pi^4 r^4$ at k are negligibly small and the squared portions in $b/\pi r$ amount to about 12 percent of $(n^2 + \beta^2)^4$ for $n = 1$. For larger values of n these terms on both sides of equation (33) become relatively still smaller. Considering that the extensional part of the deformation $(1 + \nu^2)\beta^4$ on the left side of equation (33) and the bending portion with

$k \frac{\pi^4 r^4}{b^4}$ for $n = 1$ are approximately equal, as will be

confirmed in the evaluation for the case of pure shear, there is found for the term with $b^2/\pi^2 r^2$ in the coefficient of k only about -6 percent of the no-load terms. Hence, for pure shear load ($q_2 = 0$) the terms with $b^2/\pi^2 r^2$, since they are subtracted on both sides of equation

(33), may be neglected even for $\frac{b}{\pi r} = \frac{1}{4}$.

In the coefficients of the longitudinal stress q_2 on the left side of equation (33) the squared terms in

$b/\pi r$ for $b/\pi r = \frac{1}{4}$ amount to about +6 percent of $(n^2 + \beta^2)^2$

for $n = 1$. For pure compression the dropping of the terms with $b^2/\pi^2 r^2$ and $b^4/\pi^4 r^4$ in the coefficients of k and q_2 , since they occur with different signs, would lead to an error for the buckling stress of about +13 percent. In the table given below the error, which would arise from neglecting the above-mentioned terms, are given as a function of $b/\pi r$ for $k = 1/12 s^2/r^2 = 10^{-5}$ and $\nu = 1/6$ ¹ (the exact values are taken from W. Flügge):

| $\frac{b}{\pi r}$ | $\frac{1}{10}$ | $\frac{1}{8}$ | $\frac{1}{6}$ | $\frac{1}{5}$ | $\frac{1}{4}$ | $\frac{1}{3}$ | $\frac{1}{2}$ |
|-------------------|----------------|---------------|---------------|---------------|---------------|---------------|---------------|
| Error in percent | 1.4 | 3.2 | 6 | 8 | 13 | 25 | 64 |

Allowing an error of 6 percent, the simplified computation may be employed up to central angles of 30° ($b/\pi r = 1/6$) for the case of pure compression.

Under the restrictions of 45° for the central angle for pure shear and 30° for pure compression, there is obtained from equation (33) after division by

$k \frac{\pi^4 r^4}{b^4} (n^2 + \beta^2)^2 \frac{32}{\pi} n \beta$ the following system of equations:

$$(C_n n) \frac{\pi}{32 \beta} \left[(n^2 + \beta^2)^2 + w \frac{\beta^4}{(n^2 + \beta^2)^2} + \frac{\sigma}{p^*} 4 \beta^2 \right] \frac{1}{n^2} - \frac{\tau}{p^*} \sum_m (C_m m) \frac{1}{n^2 - m^2} (-1)^m = 0 \quad (35)$$

where

$$w = 12 (1 - \nu^2) \frac{b^4}{\pi^4 r^4} \frac{r^2}{s^2} \quad (36)$$

¹The buckling formulas with the above neglected terms for the case of pure compression become the following. (See reference 1.)

$$-\frac{\sigma}{p^*} = 1 + \frac{w}{16} \quad \text{for } w \leq 16$$

$$-\frac{\sigma}{p^*} = \frac{1}{2} \sqrt{w} \quad \text{for } w \geq 16$$

The two parameters of a curved plate strip $b/\pi r$ and b/s still remain only in w and in the reference magnitude p^* . From this it may be seen that the critical stress in the case of pure shear or pure compression may each be represented by a curve for a plate strip with small curvature and in the case of combined stress by a single parameter family of curves.

If, for brevity, we set

$$T_n = \frac{\pi}{32 \beta} \left[(n^2 + \beta^2)^2 + w \frac{\beta^4}{(n^2 + \beta^2)^2} + \frac{\sigma}{p^*} 4 \beta^2 \right] \frac{1}{n^2} \quad (37)$$

then for a given σ/p^* the critical shearing stress is given by setting equal to zero the following determinant:

$$\Delta = \begin{vmatrix} T_1 & \frac{1}{3} \frac{\tau}{p^*} & 0 & \frac{1}{15} \frac{\tau}{p^*} & 0 & \dots \\ \frac{1}{3} \frac{\tau}{p^*} & T_2 & -\frac{1}{5} \frac{\tau}{p^*} & 0 & -\frac{1}{21} \frac{\tau}{p^*} & \dots \\ 0 & -\frac{1}{5} \frac{\tau}{p^*} & T_3 & \frac{1}{7} \frac{\tau}{p^*} & 0 & \dots \\ \frac{1}{15} \frac{\tau}{p^*} & 0 & \frac{1}{7} \frac{\tau}{p^*} & T_4 & -\frac{1}{9} \frac{\tau}{p^*} & \dots \\ 0 & -\frac{1}{21} \frac{\tau}{p^*} & 0 & -\frac{1}{9} \frac{\tau}{p^*} & T_5 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} = 0 \quad (38)$$

The dropping of the terms with $b^2/\pi^2 r^2$ and $b^4/\pi^4 r^4$ in the brackets after $k \frac{\pi^4 r^4}{b^4}$ in equation (33) amounts

to neglecting the trapezoidal shape of the plate element in cross section. From equations (5.1-3) and (7.1-8) there are then obtained the following simplified equations for the plate strains and for the section forces:

$$\left. \begin{aligned} \epsilon_x &= u_x - z w_{xx} \\ \epsilon_y &= v_y + \frac{w}{r} - z w_{yy} \\ \gamma_{xy} &= u_y + v_x - 2z w_{xy} \end{aligned} \right\} \text{--- (39.1-3)}$$

$$\left. \begin{aligned} N_x &= D \left(u_x + v v_y + v \frac{w}{r} \right) \\ N_y &= D \left(v u_x + v_y + \frac{w}{r} \right) \\ N_{xy} &= N_{yx} = D \frac{1-v}{2} (u_y + v_x) \\ M_x &= K(w_{xx} + v w_{yy}) \\ M_y &= K(v w_{xx} + w_{yy}) \\ M_{xy} &= M_{yx} = K(1-v)w_{xy} \end{aligned} \right\} \text{--- (40.1-6)}$$

The curvature of the plate element in these magnitudes is expressed only through the term w/r . Of the load terms there remain after striking out the portions with

$\frac{b^2}{\pi^2 r^2}$ the buckling term $-2 q_1 r w_{xy}$ and $-q_2 r w_{xx}$ in equation (15.3) and $-q_1 \beta \frac{8}{\pi} \sum_m C_m \frac{n m}{n^2 - m^2} (-1)^m$ and $q_2 \frac{\pi^2 r^2}{b^2} \beta^2$ in equation (24.3), i.e., only the buckling

terms which would remain in passing to the flat plate. In the equilibrium condition there is added to the portions for the flat plate the radial components of the ring stresses. The system of differential equations (15.1-3) now assumes the following form:

$$\left. \begin{aligned} u_{xx} + \frac{1-v}{2} u_{yy} + \frac{1+v}{2} v_{xy} + v \frac{w_x}{r} &= 0 \\ v_{yy} + \frac{1-v}{2} v_{xx} + \frac{1+v}{2} u_{xy} + \frac{w_y}{r} &= 0 \\ \frac{1}{r} \left(v u_x + v_y + \frac{w}{r} \right) + k r^2 \Delta \Delta w - 2 q_1 w_{xy} - q_2 w_{xx} &= 0 \end{aligned} \right\} \text{--- (41.1-3)}$$

By introducing a stress function, equations (40.₁₋₂) can be further combined into one. The equilibrium condition of the forces in the tangent plane to the plate element are now simply

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0$$

These equations are exactly satisfied by the expression for the stress function

$$\frac{1}{s} N_x = \Phi_{yy}, \quad \frac{1}{s} N_y = \Phi_{xx}, \quad \frac{1}{s} N_{xy} = -\Phi_{xy}$$

From equations (40) there is obtained by elimination of the displacements u and v the relation between the stress function Φ and the radial displacement w

$$\frac{r}{E} \Delta \Delta \Phi = w_{xx} \quad (42.1)$$

The differential equation of (41.₃) assumes the form

$$\frac{s}{D r} \Phi_{xx} + k r^2 \Delta \Delta w - 2 q_1 w_{xy} - q_2 w_{xx} = 0 \quad (42.2)$$

The system of differential equations (41.₁₋₃) or (42.₁₋₂), the validity of which was proved here for a central angle of 45° for the case of pure shear and about 30° for the case of pure compression, is the starting point in the investigation of Leggett.

IV. EVALUATION OF THE BUCKLING CONDITION AND SETTING-UP OF AN APPROXIMATION FORMULA FOR THE CASE OF PURE SHEAR

In the diagonal terms of the buckling determinant (38), the longitudinal stress σ_x is to be set equal to zero. Using in the expression values for n up to

$n = 2, 3 \dots$, etc., there are obtained from equations (38) for the determination of the critical shear stress the following approximation formulas:

First approximation:

$$\left(\frac{\tau}{p^*}\right)^2 = 9 T_1 T_2 \quad (43)$$

Second approximation:

$$\left(\frac{\tau}{p^*}\right)^2 = \frac{T_1 T_2 T_3}{\frac{1}{5^2} T_1 + \frac{1}{3^2} T_3} \quad (44)$$

Third approximation:

$$\begin{aligned} \left(\frac{\tau}{p^*}\right)^4 \left(\frac{1}{3 \times 7} + \frac{1}{5 \times 15}\right)^2 - \left(\frac{\tau}{p^*}\right)^2 \left[\frac{T_3 T_4}{3^2} + \frac{T_1 T_4}{5^2} + \frac{T_1 T_2}{7^2} + \frac{T_2 T_3}{15^2} \right] \\ + T_1 T_2 T_3 T_4 = 0 \end{aligned} \quad (45)$$

The further approximations need not be written out since the critical shear stresses hardly differ in the second and third approximation. In the diagonal terms T_n there enters the ratio of the plate width to the plate wave length $\beta = b/l$ as an unknown. The latter must be so chosen as to give the least critical stresses.

In figures 6 and 7 the results of the computations

are plotted against $\sqrt[4]{\omega} = \sqrt[4]{12(1 - \nu^2)} \cdot \frac{b}{\pi r} \sqrt{\frac{r}{s}}$. Keeping

the ratio b/s fixed, the curves show the dependence of the critical shear stress and the reciprocal wave length on the plate curvature. The first approximation for the critical shear stresses gives in the entire range values about 5 percent too high while the second and third practically coincide. For $\omega = 0 (r = \infty)$, there is already obtained from the third approximation the exactly computed value $\tau/p^* = 1.334$ of Southwell and Skan (reference 8). Figures 6 and 7 also contain the curves computed by Leggett for the other limiting case of a plate strip with flexurally rigid edge stiffeners ($\nu = 0$). The critical shear stresses lie considerably higher than in our case of a plate strip with longitudinal stiffeners flexurally yield-

ing in the circumferential direction ($N_y = 0$). In practical applications the edge condition will lie between these two limiting conditions and the assumption here made is on the safe side.

From the consideration of equations (44) and (45) it may be seen that on dropping all terms except those with 3^2 in the denominator there is again obtained the first approximation (43). Since the latter differs from the further ones by only 5 percent the other terms in the second, third, etc., approximations which are obtained using greater values of n in the assumed expressions should be very small. There is thus proved the statement made in section III that the errors made with increasing n in the elimination process are insignificant in the final result.

As may be seen from figure 6, the critical-stress curve with increasing w approaches a straight line asymptotically. The equation of this asymptote may be easily derived from the second approximation by introducing a new unknown and striking out small terms. The individual terms in equation (44) are the following:

$$T_1 = \frac{\pi}{32 \beta} \left[(1 + \beta^2)^2 + w \frac{\beta^4}{(1 + \beta^2)^2} \right]$$

$$T_2 = \frac{\pi}{32 \beta} \left[(4 + \beta^2)^2 + w \frac{\beta^4}{(4 + \beta^2)^2} \right] \frac{1}{4}$$

$$T_3 = \frac{\pi}{32 \beta} \left[(9 + \beta^2)^2 + w \frac{\beta^4}{(9 + \beta^2)^2} \right] \frac{1}{9}$$

As a new variable there is taken the ratio of the bending portion $(1 + \beta^2)^2$ and the extensional portion $w \frac{\beta^4}{(1 + \beta^2)^2}$ in T_1 and set equal to z^4 .

$$(1 + \beta^2)^2 = z^4 w \frac{\beta^4}{(1 + \beta^2)^2}$$

After some computation, there is obtained

$$\beta^2 - z \sqrt[4]{\omega} \beta + 1 = 0$$

As is easily shown by the numerical computation, for a large value of ω the small root must be taken for β . Neglecting small terms, there is obtained

$$\left. \begin{aligned} \beta &= \frac{1}{z \sqrt[4]{\omega}} \\ T_1 &= \frac{\pi}{32} \sqrt[4]{\omega} z \left(1 + \frac{1}{z^4} \right) \\ T_2 &= \frac{\pi}{32} \sqrt[4]{\omega} 4z, \quad T_3 = \frac{\pi}{32} \sqrt[4]{\omega} 9z, \text{ etc.} \end{aligned} \right\} \text{--- (46)}$$

and substituting in equation (44)

$$\left(\frac{\tau}{p^*} \right)^2 = \left(\frac{30 \pi}{32} \sqrt[4]{\omega} \right)^2 \frac{z^2 \left(1 + \frac{1}{z^4} \right)}{26 + \frac{1}{z^4}}$$

From the minimum condition $\frac{d(\tau/p^*)^2}{d(z^2)} = 0$, there follows

$$z = 0.956$$

$$\beta = \frac{1.045}{\sqrt[4]{\omega}} \quad (47)$$

$$\frac{\tau}{p^*} = 0.80 \sqrt[4]{\omega} \quad (48)$$

The product $\frac{\tau}{p^*} \beta$ remains for large value of ω constant and the extensional and bending portions in T_1 on account of $z = 0.956$ are almost equal.

From figure 6 it may be seen that the curve of critical stress may be replaced by its asymptote for

$\sqrt[4]{\omega} = 2.5$, i.e., for $\omega \approx 40$ with sufficient accuracy. The equation of the asymptote (48) for $\omega > 40$ may be assumed as the approximation formula for the computation of the critical shear stress. After some computation there is obtained the following buckling formula

$$\left. \begin{aligned} \tau &= 1.56 \frac{E}{(1 - \nu^2)^{3/4}} \frac{s}{b} \sqrt{\frac{s}{r}} \\ \text{hence with } \nu &= 0.3: \\ \tau &= 1.67 E \frac{s}{b} \sqrt{\frac{s}{r}} \end{aligned} \right\} \quad \text{for } \frac{b}{s} \sqrt{\frac{s}{r}} > 4.3 \quad (49)$$

For $\frac{b}{s} \sqrt{\frac{s}{r}} < 4.3$ or $\omega = 12(1 - \nu^2) \frac{b^4}{\pi^4 r^4} \frac{r^2}{s^2} < 40$, the

critical shear stresses must be taken from the curve in figure 6.

V. EXAMPLES

The accuracy of the elimination process described in section III for a central angle of 45°

$\left(\frac{b}{\pi r} = \frac{1}{4}\right)$, which was assumed as the limit for the computation carried out here, will be tested on some examples. There will first be considered a plate strip with $b/s = 50$. For $\nu = 0.3$ there is obtained from equation (36) $\omega = 172.5$ and further for $\beta^2 = 0.1$, $p^* = 2.95$. (See fig. 6.) From the buckling determinant (38) the coefficients in the expression are computed to be

$$C_2 = -0.413 C_1 \quad C_3 = -0.0556 C_1 \quad C_4 = -0.0065 C_1$$

The buckling function then assumes the following form:

$$\begin{aligned} \frac{1}{C_1} w &= \sin \frac{\sqrt{0.1} \pi x}{b} \left[-0.413 \sin \frac{2 \pi y}{b} - 0.0065 \sin \frac{4 \pi y}{b} \right] \\ &+ \cos \frac{\sqrt{0.1} \pi x}{b} \left[\sin \frac{\pi y}{b} - 0.0556 \sin \frac{3 \pi y}{b} \right] \end{aligned}$$

In figure 8 the corresponding buckling surface is represented by means of contour lines. In figure 9 this is compared with the buckling surface of a flat plate taken from the paper by Southwell and Skan (reference 8). From a comparison of the two figures it may be seen that the wave length of the curved plate strip is larger and the folding angle made with the longitudinal axis smaller than for the flat plate.

The accuracy of the first approximations for A_n and B_n (equation (29)) depends on the ratios of the neglected portions in the coefficients of A_n , B_n , and C_n , to the remaining portions a_n , b_n , etc., in equations (31.1-3). For the absolute values of the ratios in equation (31.1) there is obtained

$$\text{for } n = 1: 0.000615$$

$$n = 2: 0.0145$$

$$n = 3: 0.0198$$

$$n = 4: 0.347$$

The ratios in equations (31.2) and (31.3) are, as may also be seen from the form of the equations, somewhat greater for $n = 1$ than for the equation (31.1) but for $n = 2$ to 4, smaller. From the computed values it may be seen that the first approximations for A_n and B_n for $n = 1$ to 3 are sufficiently accurate. The larger error for $n = 4$ is unimportant for the computation of the critical stresses on account of the small magnitude of the corresponding load terms.

For larger values of w (i.e., for larger b/s at the same central angle of 45°) the equations (46), (47), and (48) may be substituted for the values T_n , β , and τ/p^* , corresponding to the good agreement of the asymptote to the second approximation with the exact τ/p^*

curve. Since $\sqrt[4]{w}$ enters as a factor in all these equations, there is obtained from the buckling determinant (38) independently of the value of w , the following values for the coefficients in the expression for w :

$$C_2 = -0.380 C_1 \quad C_3 = -0.0470 C_1 \quad C_4 = -0.00565 C_1$$

From the comparison with the accurately computed coefficients for $\omega = 172.5$, it is seen that the buckling shape changes only slightly in the y direction for larger values of ω and is essentially distorted only in the x direction, corresponding to equation (47) for the reciprocal wave length.

The ratios in equation (31.1) for $\omega = 172.5$ become at larger values of ω and same central angles still smaller on account of the small change in the coefficients for the w expression with decreasing s^2/b^2 .

VI. SUMMARY

In the present paper the buckling condition for an infinitely long plate strip with constant curvature and simply supported longitudinal edges is set up for combined shear and longitudinal stress. Starting from the complete fundamental equations for the displacements a simple computation process is developed for plate strips of small curvature (central angle $\leq 45^\circ$) with the aid of a Ritz expression. For the case of pure shear stress the

buckling condition is evaluated and for $\frac{b}{s} \sqrt{\frac{s}{r}} > 4.3$ ($\omega > 40$) a simple approximation formula derived. (See fig. 6 and equation (49).) The computation results for the general case of combined shear and axial stresses will be reported on later.

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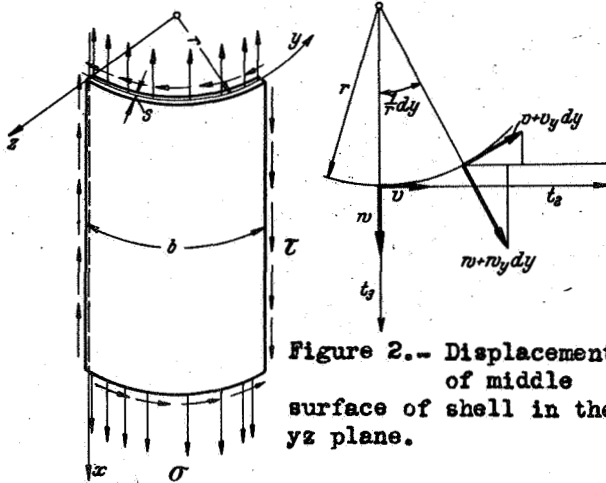


Figure 2.-- Displacement of middle surface of shell in the yz plane.

| Gl. | n | C_{1n} | | | | | | C_{2n} | | | | | |
|--------|---|----------|---|---|---|---|---|----------|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 | 6 | 1 | 2 | 3 | 4 | 5 | 6 |
| (21.1) | 1 | — | — | — | — | — | — | — | — | — | — | — | — |
| | 2 | — | — | — | — | — | — | — | — | — | — | — | — |
| | 3 | — | — | — | — | — | — | — | — | — | — | — | — |
| | 4 | — | — | — | — | — | — | — | — | — | — | — | — |
| | 5 | — | — | — | — | — | — | — | — | — | — | — | — |
| | 6 | — | — | — | — | — | — | — | — | — | — | — | — |
| (21.2) | 1 | — | — | — | — | — | — | — | — | — | — | — | — |
| | 2 | — | — | — | — | — | — | — | — | — | — | — | — |
| | 3 | — | — | — | — | — | — | — | — | — | — | — | — |
| | 4 | — | — | — | — | — | — | — | — | — | — | — | — |
| | 5 | — | — | — | — | — | — | — | — | — | — | — | — |
| | 6 | — | — | — | — | — | — | — | — | — | — | — | — |

Figure 5.-- Scheme of coefficients of unknowns C_{1n} , C_{2n} in equation (21).

Figure 1.-- Plate strip under combined shear and axial stress.

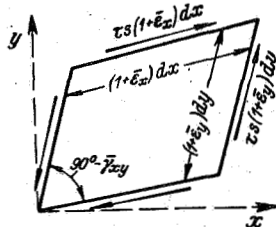


Figure 4.-- Shear forces on deformed plate element.

Figure 6.-- Critical shear stress of a curved plate strip as a function of

$$\sqrt{\omega} = \sqrt{\frac{12(1-\nu^2)}{\pi^2} \frac{b}{\pi r} \sqrt{\frac{r}{s}}} \\ (p^* = \frac{\pi^2 E s^3}{3(1-\nu^2) b^3})$$

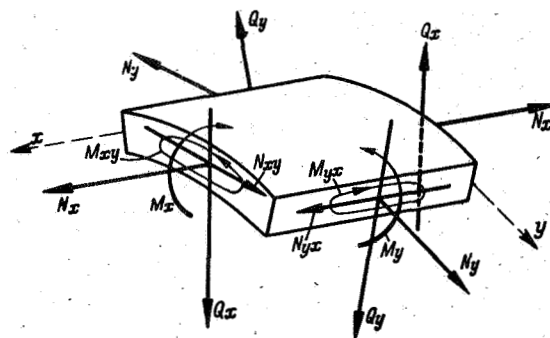


Figure 3.-- Section forces on curved plate element.

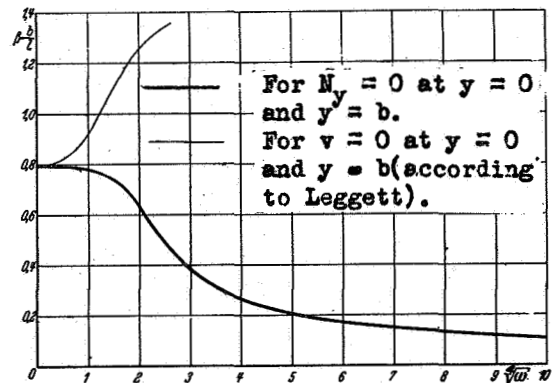
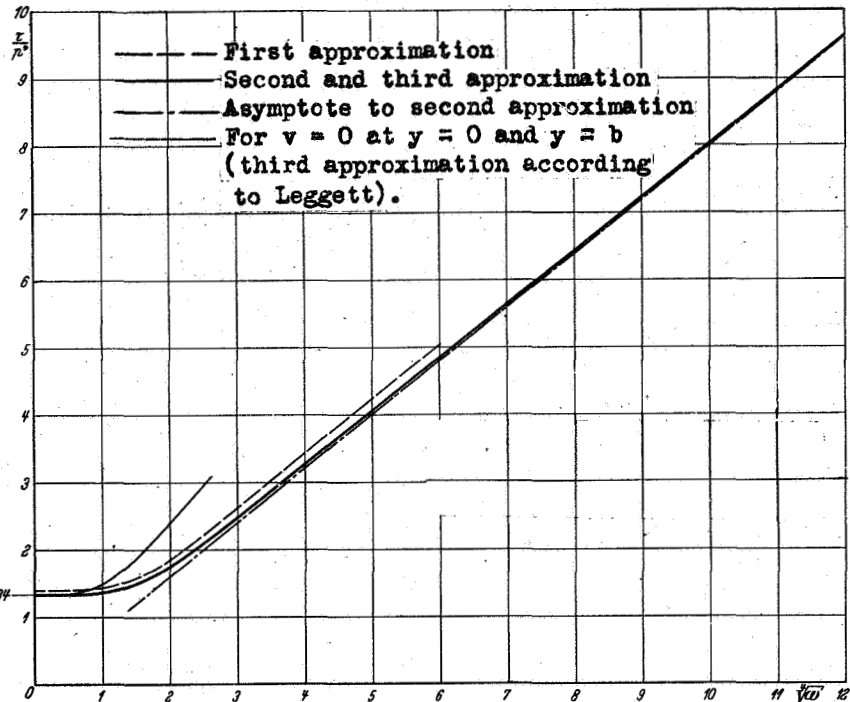


Figure 7.-- The reciprocal wave length β for the smallest critical shear stresses as a function of $\sqrt{\omega}$